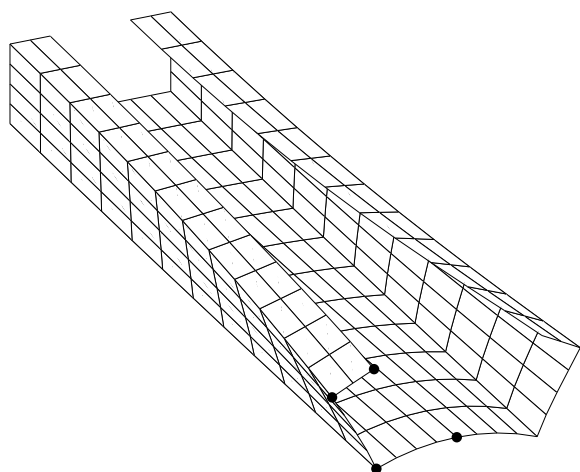


# Advanced Beam Elements with Distorting Cross Sections



Martin Mygind, s100041

**MSc Thesis**

Professor and Supervisor: Jeppe Jönsson

Department of Civil Engineering  
2013

DTU Civil Engineering  
February 2013

## CONTENTS

### Contents

<b>Preface</b>	<b>i</b>
<b>Abstract</b>	<b>ii</b>
<b>Resumé</b>	<b>iii</b>
<b>Nomenclature</b>	<b>iv</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Aim of Work . . . . .	1
1.2 Methodology . . . . .	1
1.3 Benefits of GBT . . . . .	1
<b>2 Kinematics and Constitutive Relations</b>	<b>2</b>
2.1 Coordinate system . . . . .	2
2.2 Kinematic Assumptions - Displacement Field . . . . .	2
2.3 Constitutive Relations . . . . .	6
2.3.1 Without Poisson Effect . . . . .	6
2.3.2 With Poisson Effect . . . . .	6
<b>3 Governing Homogeneous Differential Equations</b>	<b>7</b>
3.1 Potential Energy . . . . .	7
3.1.1 Without Poisson Effect . . . . .	7
3.1.2 With Poisson Effect . . . . .	8
3.2 Interpolation Functions . . . . .	9
3.3 Local Stiffness Matrices . . . . .	10
3.4 Global Stiffness Matrices . . . . .	10
3.4.1 Transformation between Local Direction and Global Direction	11
3.4.2 Assembly of Global Stiffnesss Matrices . . . . .	12
3.5 Governing Differential Equations . . . . .	12
3.5.1 Without Poisson Effect . . . . .	13
3.5.2 With Poisson Effect . . . . .	14
3.6 Inner Equilibrium Equations . . . . .	16
3.7 Sub-Conclusion regarding the Inclusion of the Poisson Effect . . . . .	17
3.8 Main Stiffness Terms . . . . .	17
3.8.1 Cholesky Decomposition Theory . . . . .	18
3.9 Transformation of Matrix Formulation . . . . .	19
<b>4 Identification of Natural Beam Modes</b>	<b>21</b>
4.1 Step 1 - Transformation, Elimination and Reduction of Order . . . . .	21
4.2 Step 2 - Pure Orthogonal Axial Mode . . . . .	23
4.2.1 Step 2 - Considerations on the Axial Mode . . . . .	24

## CONTENTS

4.3	Step 3 - Rigid Cross Section Displacement - Orthogonal Bending Modes	24
4.4	Step 4 - Rigid Cross Section Displacement - Orthogonal Rotation Mode	25
4.5	Step 5 - Identification of the Two Unknown Orthogonal Modes . . . . .	26
4.5.1	Assumed Shear . . . . .	26
4.5.2	Assumed Bending . . . . .	27
4.6	Step 6 - Verification of Identified Solutions $\lambda = 0$ . . . . .	28
4.7	Step 7 - Back-Substitution . . . . .	28
4.8	Step 8 - Verification of All Found Solutions . . . . .	29
4.9	Considerations on the Solutions from the Elimination of $(\psi \mathbf{V}_\Omega^o)''$ . . . . .	30
4.9.1	Decoupling of Modes Corresponding to Elimination of $(\psi V_\Omega^o)''$ , Approach 1 . . . . .	31
4.9.2	Decoupling of Modes Corresponding to Elimination of $(\psi V_\Omega^o)''$ , Approach 2 . . . . .	32
4.10	Step 9 - Normalization of Found Solutions . . . . .	33
4.11	Main Points from Section 4 . . . . .	34
<b>5</b>	<b>Mode Shapes</b>	<b>35</b>
5.1	Closed Profile . . . . .	36
5.2	Open Profile . . . . .	38
5.3	Comments on Mode Shape Plots . . . . .	40
<b>6</b>	<b>Axial Variation Functions</b>	<b>41</b>
6.1	Conventional Beam Deformation Modes . . . . .	41
6.1.1	Pure Axial Mode . . . . .	41
6.1.2	Bending Modes . . . . .	41
6.1.3	Last Two Modes . . . . .	42
6.1.4	Rotational Mode . . . . .	42
6.1.5	Solution Constants for Identified Forms . . . . .	43
6.2	Homogeneous Axial Variation Functions . . . . .	43
6.2.1	Pure Axial Mode . . . . .	43
6.2.2	Bending Modes . . . . .	44
6.2.3	Rotational Mode . . . . .	45
6.2.4	Remaining Modes . . . . .	47
<b>7</b>	<b>Beam Element Stiffness Matrix</b>	<b>48</b>
7.1	General Beam Element Stiffness Formulation . . . . .	48
7.2	Optimized Beam Element Stiffness Formulation . . . . .	49
7.3	Numerical Construction of Integral Part of Beam Stiffness Matrix . . . . .	51
7.4	Numerical Solution of Integral Terms . . . . .	53
7.4.1	Part 1 . . . . .	55
7.4.2	Part 2 . . . . .	55
7.4.3	Part 3a . . . . .	56
7.4.4	Part 3b . . . . .	56

## CONTENTS

7.4.5	Part 3c . . . . .	56
7.4.6	Part 4 . . . . .	57
<b>8</b>	<b>Generalized Displacement and Boundary Conditions</b>	<b>58</b>
<b>9</b>	<b>Comparative Premises</b>	<b>61</b>
<b>10</b>	<b>Deformation Test of a Single GBT Beam Element</b>	<b>63</b>
10.1	Load through Boundary Conditions . . . . .	63
10.2	Beam Element Length . . . . .	64
10.3	Abaqus Parameters . . . . .	64
10.4	Load Combinations and GBT Deformation Plots . . . . .	66
10.5	Deformation Results . . . . .	72
10.6	Comments on Results . . . . .	73
<b>11</b>	<b>Deformation and Stress Test of Two Assembled GBT Beam Elements</b>	<b>75</b>
11.1	Finite Element Formulation . . . . .	75
11.2	Calculation of Stresses . . . . .	76
11.3	Abaqus Parameters . . . . .	77
11.4	Load Combinations and GBT Deformation Plots . . . . .	78
11.5	Deformation Results . . . . .	80
11.6	Comments on Results . . . . .	80
11.7	GBT Stress Plots . . . . .	80
11.7.1	Closed Profile . . . . .	82
11.7.2	Open Profile . . . . .	83
11.7.3	Closed Profile - Convergence Test . . . . .	84
11.8	Stress Results . . . . .	85
11.9	Comments on Results . . . . .	85
<b>12</b>	<b>Membrane and Bending Stresses</b>	<b>87</b>
12.1	Axial Stresses, $\sigma$ . . . . .	88
12.2	Transverse Stresses, $\sigma_s$ . . . . .	89
12.3	Shear Stresses, $\tau_{sz}$ . . . . .	90
12.4	Comments on Results . . . . .	91
<b>13</b>	<b>Conclusion</b>	<b>92</b>
<b>14</b>	<b>Future Work</b>	<b>94</b>
<b>15</b>	<b>References</b>	<b>96</b>

CONTENTS

<b>16 Appendices</b>	<b>97</b>
Appendix 1 - Element Stiffness Matrices . . . . .	98
Appendix 2 - Derivation of Governing Homogeneous Differential Equations Including the Poisson Effect . . . . .	99
Appendix 3 - Investigation of Governing Differential Equations Including the Poisson Effect . . . . .	106
Appendix 4 - Investigation of Governing Differential Equations Without Poissons Effect . . . . .	114
Appendix 5 - Inner Equilibrium Equations . . . . .	121
Appendix 6 - Example of Transformation Matrix . . . . .	129
Appendix 7 - Stiffness Matrices from Maple . . . . .	130
Appendix 8 - Assembly of $\Psi\mathbf{J}\Psi$ in Maple . . . . .	135
Appendix 9 - Differentiated Interpolation Functions from Maple . . . . .	141
Appendix 10 - Hand Calculations from Mathcad . . . . .	143
Appendix 11 - HTML Visualisation of Matlab Main File . . . . .	145
<b>17 Enclosed CD-ROM</b>	<b>151</b>

## Preface

This Master's Thesis is a part of a research project on distortional behaviour of thin-walled structural elements lead by Professor Jeppe Jönsson and Associate professor Michael Joachim Andreassen at the Department of Civil Engineering at DTU. The research has been performed in cooperation with Lotte Braad Sander while the reporting part has been handled individually. The Matlab program and manual have been developed jointly by both parties.

The thesis primarily relates to the work done in the articles [Jönsson & Andreassen, 2010] and [Jönsson & Andreassen, 2012]. References to these articles are therefore as a main rule avoided except for citations or when it is found necessary to emphasize the references. Parts of the work performed by previous student Michael Teilmann Nielsen in his Master's Thesis from 2012 is also further elaborated.

In order to get an idea of how the *Generalized Beam Theory* (GBT) is implemented in Matlab, a HTML visualisation of the main Matlab file has been added in Appendix 11. Unfortunately the HTML visualisation is at the expense of vector graphics, but the clarity of the script is preferable.

This Master's Thesis comprises 30 ECTS credits and has been submitted on February 22, 2013.

*Kgs. Lyngby  
February 2013  
Martin Mygind*

## Acknowledgements

Jeppe Jönsson

*Professor and Head of Section at the Department of Civil Engineering at DTU*

For guidance and supervision in the research process. The time used both during and after working hours is greatly appreciated.

Michael Joachim Andreassen

*Assistant Professor at the Department of Civil Engineering at DTU*

For the many excellent advices on the programming part.

## Abstract

The use of thin-walled members is increasing as are the attention and need for more detailed calculations comprising the distortional behaviour of thin-walled members. In order to include the distortional behaviour in the calculations it is necessary to extend the classic Euler-Bernoulli theory. The formulation and implementation of an advanced beam element in a finite element context which facilitate both flexural, torsional and distortional behaviour is considered. The formulation of the beam element relates to the work performed in [Jönsson & Andreasen, 2010] and [Jönsson & Andreasen, 2012]. Thus this Master's Thesis consists of; a Matlab program based on *generalized beam theory* (GBT), a manual for the program and the present thesis on the theory and research from which the Matlab program is developed.

The formulation and development of the advanced beam element is based on a cross-sectional analysis, where the cross-section is assembled from straight beam elements which hold both in-plane and one out-of-plane degrees of freedom. A displacement field based on Bernoulli theory is introduced and the potential energy is established through simple constitutive relations. It is highly desirable to include *Possion* in the constitutive relations but the inclusion produces a coupling term which complicates the GBT-formulation wherefore the coupling term is neglected.

From a virtual work formulation the first variation of the potential energy is investigated by the introduction of a virtual displacement field and by means of partial integration leading to the governing differential equations (GDE). It is found that the GDE can also be derived from inner equilibrium equations leading to two sets of GDEs, which when added are identical to the ones found from the virtual work formulation.

From the GDE a generalized eigenvalue problem is constructed from which the conventional beam and distortional deformation modes are identified. The axial variation functions for each cross-sectional deformation mode are identified and special attention is given to the conventional beam deformation modes and the related axial variation functions. Hence the theory is denominated semi-discretization as the cross-section is discretized while the solution along the beam is represented by analytical functions. Based on the axial variation functions and the cross-sectional deformation modes, the distorting beam element formulation is developed.

Displacement and stresses are determined using general finite element (FE) theory. Due to the chosen displacement field and interpolation functions it is possible to split the stresses into a membrane part and a bending part. Deformation and stress tests are performed and the results are compared to results obtained from [Nielsen, 2012] and the commercial FE-program Abaqus whereupon conclusions are drawn on the basis of the performance of the beam element.

## Resumé

Brugen af tyndvæggede profiler øges, ligesom behovet for mere detaljerede beregninger, som omfatter forvrængningen/formændringen af de tyndvæggede profiler stiger. For at kunne inkludere forvrængningsdeformationen i beregningerne er det nødvendigt at udvide den klassiske Euler-Bernouille teori. I dette speciale udvikles og implementeres et avanceret bjælkeelement i en finite element kontekst, hvor både bøjning, rotation og forvrængning er mulig. Konstruktionen af bjælkeelementet relaterer sig til arbejdet udført i [Jönsson & Andreasen, 2010] og [Jönsson & Andreasen, 2012]. Følgelig består dette kandidatspeciale af; et Matlab program baseret på en generaliseret bjælke teori (GBT), en manual for programmet og nærværende afhandling, som indeholder teori og forskning, hvorfra Matlab programmet er konstrueret.

Formuleringen og udviklingen af det avancerede bjælkeelement bygger på en tværsnit-analyse, hvor tværsnittet er opbygget af lige bjælkeelementer, som har både frihedsgrader i planen og ud af planen. Et flytningsfelt baseret på Bernoulli-teori introduceres, og den potentielle energi kan bestemmes gennem simple konstitutive relationer. Som udgangspunkt er det forsøgt at inkludere *Possion* i de konstitutive ligninger, men dette medfører et koblingsled, som besværliggør GBT-formuleringen, hvorfor der ses bort fra dette koblingsled.

Ved brug af virtuelt arbejdes princip undersøges den første variation af den potentielle energi ved at introducere et virtuelt flytningsfelt samt partiel integration, hvorved de styrende differentiaalligninger (SDL) findes. Ydermere er det fundet, at SDL kan findes ud fra indre ligevægtsligninger, hvor disse SDL er identiske med dem, der er fundet ved brug af virtuelt arbejdes princip.

Fra SDL konstrueres et generaliseret egenværdiproblem, hvorved de konventionelle bjælke og forvrængede formfunktioner kan bestemmes. Efterfølgende bestemmes de respektive aksiale løsningsfunktioner, relateret til tværsnittets formfunktioner, og der sættes særlig fokus på identifikationen af de konventionelle bjælke-formfunktioner og de relaterede aksiale løsningsfunktioner. Følgelig benævnes teorien semi-diskretisering da tværsnittet er diskretiseret mens løsningen langs bjælkeelementet beskrives ved analytiske funktioner. Formuleringen af bjælkeelementet sker gennem tværsnittets formfunktioner og de aksiale løsningsfunktioner.

Deformationer og spændinger bestemmes ved brug af generel finite element (FE) teori, og grundet det valgte flytningsfelt og de valgte interpolationsfunktioner, er det muligt at opdele spændingerne i en membrandel og en bøjningsdel. Deformations- og spændingstest udføres, og resultaterne sammenlignes med resultater fra [Nielsen, 2012] og resultater fra the kommercielle FE-program Abaqus, og der konkluderes på bjælkeelementets præstation.